

# On a physical realization of Chern-Simons Theory

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## **Abstract**

The physical content of Chern-Simons-action is discussed and it is shown that this action is proportional to the usual charged matter interaction term in electrodynamics.

During the last decade we had a rapid success of Chern-Simons theory in theoretical physics [1]. Almost in every part of physics where the topological and low dimensional effects play roles, one finds applications of this theory [2]. This applications show the physical relevance of Chern-Simons theory in a clear manner.

However, there are still two open questions about the physical realization of this theory, namely about:

1. A physical realization of Chern-Simons theory according to the well known and observable physical phenomena [3].
2. The physical relationship between the well known Maxwell/Yang-Mills theories and the Chern-Simons theory.

We discuss here the first question and show that there is indeed a well known electrodynamical realization of the U(1) Chern-Simons structure according to the Ohm's equations.

A generalization to the U(N) Chern-Simons theories can be also considered in view of the fact that the gauge  $A_0 = 0$  which is needed for quantization of the U(N) version reduces also the U(N) form into the U(1) form.

To begin let us mention an early result of us which comes out in relation with the theory of IQHE [4].

If we consider the Chern-Simons action  $S_{C-S} = \int d^3x \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma$  in the usual Weyl-gauge  $A_0 = 0$ , then the action becomes [5]

$$S = \int_M \epsilon_{mn} A_m \frac{dA_n}{dt} \quad , m, n = 1, 2 \quad , \quad (1)$$

where  $M := \Sigma \times \mathbf{R}$  is the three dimensional space-time manifold.

If we insert the Ohm's equations in two dimensions  $j_m = \sigma_H \epsilon_{nm} \frac{dA_n}{dt}$  with  $\epsilon_{mn} = -\epsilon_{nm} = 1$  which are known, as phenomenological relation or "material equations", from the quantum Hall effect with  $j_m = nev_m$  and  $\sigma_H$  as the current density and the Hall conductivity respectively, where  $n$  is the charge carrier density and  $e$  is the elementary charge. Then, the action (1) becomes proportional to the following one

$$Q \int A_m dx^m \quad , \quad (2)$$

which is well known from the classical electrodynamics for interaction term between potential and charged matter [6], where  $C = 1$  and  $Q = \int \int ne$  and the proportionality factor is the Hall resistivity.

Thus, it is obvious that one may obtain in the opposite way the Chern-Simons electrodynamics from the standard interaction term between the charged matter and electromagnetic potential in the classical action functional [6].

To arrive at the physically general relevant form one should begin with the usual interaction term  $Q \int A_\mu dx^\mu$  between a charged matter and electromagnetic potential in classical theories where  $\mu = \{0, \dots, 3\}$ .

Considering a continuous 3-D charge distribution one obtains [6]:

$$Q \int A_\mu dx^\mu = \int d^4x A_\mu j^\mu \quad , \quad (3)$$

where we used the definition of current density  $j_\mu = nev_\mu$ .

In view of the continuity equation for  $j_\mu$  or in view of the gauge dependence of electromagnetic potential we have to choose gauge fixing condition(s), e. g.  $j_0 = 0$  or  $A_0 = 0$  to arrive at physically relevant interaction term [7]. This reduces the above mathematically general invariant term to the following physical general invariant interaction term

$$\int dt \int d^3x A_\alpha j^\alpha \quad , \quad \alpha = 1, 2, 3 \quad (4)$$

Recall that , if we choose the gauge fixing condition  $j_3 = 0$  or  $A_3 = 0$ , then  $\alpha = \{0, 1, 2\}$  and  $dt$  must be replaced by  $dz$ .

Thereafter, we can rewrite it in view of Ohm's equations  $j_\alpha = \sigma_H \epsilon_{\alpha\beta\gamma} \partial_\beta A_\gamma$  in the following invariant form

$$\sigma_H \cdot \int dt \int d^3x \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \quad , \quad (5)$$

where  $\sigma_H$  is a locally constant quantity called the Hall conductivity.

The above volume integral is the well known Chern-Simons invariant in 3 or 2 + 1-D manifold. Thus, we find a very general physical realization of the Chern-Simons action functional in terms of the very usual interaction term between matter and the electromagnetic potential.

Here, the volume integral, i.e. the Chern-Simons-action can be considered as the interaction Lagrangian between charged matter and the electromagnetic potential.

If we choose for the second gauge fixing condition  $A_0 = 0$  and use  $A_m := A_m(x_m, t)$ , then the relation reduces to the one proportional to the relation (1), where the proportionality factor is a constant times  $\sigma_H$ .

Moreover, if we start with a  $2 - D$  charge distribution in the interaction term (3), then we arrive at  $Q_{(2-D)} \int A_\mu dx^\mu = \sigma_H \int d^3x \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma$ .

Thus, the question is if the dimension of charge distribution should be related with the true degrees of freedom of the electromagnetic field or of photon which is the intermediating field of electromagnetic interaction, i. e. also responsible for charged particle. This question, however should be discussed separately [8].

It is interesting to mention that the usual Weyl gauge fixing  $A_0 = 0$  which reduces the general Chern-Simons action  $\int A \wedge dA$  in  $2 + 1 - D$  to its *quantizable* form:

$$\int dt \int d^2x \epsilon_{mn} A_m \partial_t A_n \quad , \quad (6)$$

according to the  $[A_m(X, t), A_n(Y, t)] = i\hbar \epsilon_{mn} \delta^2(X - Y); X, Y \in \Sigma$  postulate [5], reduces also the number of degrees of freedom of electromagnetic field which was originally three to its physically true degrees of freedom, i. e. to two.

This is a hint about the physical relevance of the Chern-Simons action in the sense that the quantizable Chern-Simons action, i. e. action (6) depends only on the physically relevant degrees of freedom of the electromagnetic field, whereas in case of Maxwell's action there is no such relation between the quantization conditions on this action and the true degrees of freedom of electromagnetic field. The

reason is of course the different structures of action in the two cases and that the U(1) field strength is gauge invariant.

Moreover, the constraints of Chern-Simons action  $\epsilon_{mn}\partial_m A_n = 0$  forces the electromagnetic potential to be a pure gauge potential. On the other hand, it is known from the geometric quantization that the quantization is caused by the existence of a flat connection on a line bundle over the phase space [9]. In the case under consideration, it is the pure electromagnetic gauge potential which should play the role of the mentioned flat connection in geometric quantization, where the related U(1) bundle of gauge transformation, as a principal bundle, should replace the mentioned equivalent line bundle.

This conception is also in accordance with the Borel-Weil construction of the Hilbert spaces as the space of holomorphic sections of a complex line bundle  $L_\lambda$  over the  $G/T$  where  $G$ ,  $T$  and  $\lambda$  are a compact finite dimensional group, its maximal Abelian subgroup and the highest weight of an irreducible representation of  $G$  [10].

Thus, the quantized Chern-Simons action in its Schroedinger representation  $\phi = e^{i\sigma_H S_{CS}} = e^{iQ \int_C A_m dx^m}$  represents the phase change of a charged system moving in an electromagnetic potential field:

$$\Psi_c^A(x_m, t) = e^{iQ \int_C A_m dx^m} \Psi(x_m, t) \quad (7)$$

which is a path dependent gauge transformation of  $\Psi(x_m, t)$ .

The path independent phase change or gauge transformation is then given by a closed path contribution which is equivalent to the Bohm-Aharonov effect for  $B \neq 0$ , i. e. in a multiply connected region. Followingly, this could be considered as an observable physical realization of Chern-Simons-theory, if one recalls the Ohm's equation for charge carriers. A related quantum mechanical realization of Chern-Simons theory according to the above relation with  $\int A$  should be seen in the flux quantization in view of  $\oint A = \frac{h}{e}$ .

An other application of the mentioned relation between Chern-Simons-action and the usual interaction term is that one can show the equivalence between the Kubo-Thouless approach to the IQHE which is based on the linear response theory [2] and the Chern-Simons-approach to the same [4]. The Kubo's linear response theory use as perturbation term the  $H_t = A_\mu j^\mu$ . Thus, as it is shown already the perturbative

action will be in the  $2 + 1$  dimensional case, in view of Ohm's equations, equal to the Chern-Simons-action. Recall that in the linear response approach the conductivity is identified by comparison with other phenomenologically known formula, thus it is identified here phenomenologically. This is equivalent to the use of Ohm's equations as phenomenological equations, in the above discussed relation between two action terms.

This means that in case of theory of conductivity the Chern-Simons interaction term and linear response interaction term are both equivalent to the usual interaction term  $A_m dx^m = A$ .

Moreover, the related prove of antisymmetry of conductivity in IQHE case according to the linear response approach is a consequence of antisymmetry of  $F_{mn}$  in the integral relation  $\oint A = \int \int F$ , with  $F_{mn} := [D_m, D_n]$  and  $D_m = \partial_m - A_m$ . The reason lies in the flux quantization  $\oint A = \int \int F = \frac{h}{e}$  which should be responsible also for the quantization in IQHE case [8].

In conclusion let us mention that also the mentioned second question should be answered according to the above discussed point of view, if one recalls the relation between the Chern-Simons-action and the Adler-Bell-Jakiw anomaly term:

$$Tr \int dA \wedge dA = Tr \left( d \int A \wedge dA \right) \quad (8)$$

which arises in quantization of Yang-Mills action [8].

Footnotes and references

## References

- [1] For preliminaries and mathematical questions see: E. Guadagnini, The Link Invariants of the Chern-Simons Field Theory, (Walter de Gruyter, Berlin. New York 1993) and references therein.
- [2] Beyond the application of Chern-Simons Field theory in the quantum Hall Effect (IQHE and FQHE), one has also applications in superconductivity, Chaos theory, etc. (see Ref. [4] and references therein for related applications)

- [3] It could be a realization according to the classical or quantum physics. Nevertheless, it is important to mention that the classical interpretation of Chern-Simons theory according to the electromagnetic interaction term for charged matter which is discussed here, have its quantum mechanical realization, e.g. according to the Bohm-Aharonov effect.
- [4] For various applications of Chern-Simons theory in Solid state physics see:  
 F. Ghaboussi, "On the Integer Quantum Hall Effect", KN-UNI-preprint-95-1; "A Model of the Integer Quantum Hall Effect", KN-UNI-preprint-95-2, submitted for publication; See also "On The Quantum Theory of Hall Effect", KN-UNI-preprint-95-3 , submitted for publication (quant-ph/9603011); "The Edge Currents and Edge potentials in IQHE", KN-UNI-preprint-95-4, submitted for publication; "On The Relation between Quantum Hall Effect and Superconductivity", KN-UNI-preprint-95-5, submitted for publication, and references therein (quant-ph/9603013); Quantum Hall Effect and Chaotic Motion in Phase Space", KN-UNI-preprint-95-6,.
- [5] E. Witten, Cumm. Math. Phys. 121 (1989) 351-399.
- [6] See L.D.Landau, E.M.Lifschitz, II Vol.
- [7] In view of the gauge dependence of the electromagnetic potential  $A'_\mu = A_\mu + \partial_\mu \Lambda$  one has to choose two gauge fixing conditions, e.g. the radiation or Coulomb gauge, to retain the two true degrees of freedom for the electromagnetic potential or photon.
- [8] Under preparation. For the relation between flux quantization and the quantization of Hall conductivity in IQHE see the last paper in Ref. [4].
- [9] N. Woodhouse, Geometric Quatization, (Oxford University Press, 1980 and 1990)
- [10] C. Nash, Differential Topology and Quantum Field Theory, (Academic Press, 1991)